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Entropy for Zero-Temperature Limits of Gibbs-Equilibrium States for Countable-Alphabet Subshifts of Finite Type

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Let Σ_A be a finitely primitive subshift of finite type over a countable alphabet. For suitable potentials $f\colon \Sigma_A \to \mathbb{R}$ we can associate an invariant Gibbs equilibrium state μ_{tf} to the potential tf for each $t \geq 1$. In this note, we show that the entropy $h(\mu_{tf})$ converges in the limit $t \to \infty$ to the maximum entropy of those invariant measures which maximize $\int f\, d\mu$. We further show that every weak-* accumulation point of the family of measures μ_{tf} has entropy equal to this value. This answers a pair of questions posed by O. Jenkinson, R. D. Mauldin and M. Urbański.

KEY WORDS: Gibbs state, equilibrium state, ground state, maximizing measure, countable alphabet subshift of finite type, entropy

1. INTRODUCTION

Let $\sigma\colon \Sigma_A \to \Sigma_A$ be a subshift of finite type on a countable alphabet $\mathcal{I} \subset \mathbb{N}$, and let $f\colon \Sigma_A \to \mathbb{R}$ be uniformly continuous. Under suitable conditions (to be made precise below) it may be shown that f admits a unique σ-invariant Gibbs state μ_f. In this note we study the behaviour of families of invariant Gibbs states $(\mu_{tf})_{t\geq 1}$ in the limit $t \to \infty$. Our result provides a complete characterisation of the limiting behaviour of the entropy $h(\mu_{tf})$, and extends previous results known in the case where the alphabet $\mathcal{I}$ is finite. In particular, we answer two questions posed by O. Jenkinson, R. D. Mauldin and M. Urbański in a previous investigation of this topic.[(6)].

In the thermodynamic interpretation, our parameter t corresponds to an inverse temperature of a system, and the measure μ_{tf} corresponds to the equilibrium

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of the system at temperature $1/t$. The limit $t \to \infty$ is therefore a zero temperature limit, and accumulation points of the measures μ_{tf} constitute *ground states*. The aim of the present note is to show that the entropy μ_{tf} is continuous in the zero temperature limit, in the sense that $h(\mu_\infty) = \lim_{t\to\infty} h(\mu_{tf})$ whenever μ_∞ is an accumulation point at infinity of the measures μ_{tf}.

The history of results of this type in the finite-alphabet case is somewhat convoluted; results analogous to those in the present paper and in (Ref. 6) were proved independently on a number of occasions, e.g. (Refs. 1–3, and 5). The history of these and related results is discussed briefly in (Ref. 4, Sec. 5).

2. DEFINITIONS AND STATEMENT OF THEOREM

In this section we review some necessary background from the thermodynamic formalism of countable-alphabet subshifts of finite type. Our reference for this section is the work of R.D. Mauldin and M. Urbański[7] (Sec. 2) (but see also (Refs. 8,11, and 12)). We then give a formal statement of our results.

Let $\mathcal{I} \subseteq \mathbb{N}$ be a countably infinite set, and let $A: \mathcal{I} \times \mathcal{I} \to \{0, 1\}$ be a square matrix. We define the shift space Σ_A associated to A to be the set

$$\Sigma_A := \{x = (x_n)_{n\geq 1} : x_n \in \mathcal{I} \text{ and } A(x_n, x_{n+1}) = 1 \text{ for all } n \geq 1\},$$

and define the shift map $\sigma: \Sigma_A \to \Sigma_A$ by

$$\sigma[(x_n)_{n\geq 1}] = (x_{n+1})_{n\geq 1} .$$

When $(i_1, \dots, i_m) \in \mathcal{I}^m$ we define the corresponding *cylinder set* by

$$[i_1, \dots, i_m] := \{x = (x_n)_{n\geq 1} \in \Sigma_A : x_j = i_j \text{ for all } 1 \leq j \leq m\}.$$

We then define a topology on Σ_A by declaring every cylinder set to be open. Given $f: \Sigma_A \to \mathbb{R}$ and $n > 0$, define

$$\mathrm{var}_n f := \sup_{(i_1,\dots,i_n)\in\mathcal{I}^n} \sup_{x,y\in[i_1,\dots,i_n]} |f(x) - f(y)| .$$

We say that f has *summable variations* if the quantity $|f|_{\mathrm{var}} = \sum_{n=1}^{\infty} \mathrm{var}_n f$ is finite. Note that if $|f|_{\mathrm{var}} < \infty$ then f is continuous, but need not be bounded. For $n > 0$ and $x \in \Sigma_A$, we use the abbreviation $S_n f(x)$ to denote the ergodic sum $\sum_{j=0}^{n-1} f(\sigma^j x)$.

We denote the set of σ-invariant Borel probability measures on Σ_A by $\mathcal{M}_\sigma$. We equip $\mathcal{M}_\sigma$ with the following *weak-* topology*: we say that $(\mu_n)_{n\geq 1}$ converges weakly-* to μ if and only if $\int f\, d\mu_n \to \int f\, d\mu$ for every bounded continuous $f: \Sigma_A \to \mathbb{R}$. In our case, an equivalent condition is that $\mu_n(B) \to \mu(B)$ for all cylinder sets B.

Given $f\colon \Sigma_A \to \mathbb{R}$, we define the *maximum ergodic average* of f to be the quantity

$$\beta(f) = \sup \left\{ \int f \, d\nu \colon \nu \in \mathcal{M}_\sigma \right\}.$$

When $\int f \, d\mu = \beta(f)$ for $\mu \in \mathcal{M}_\sigma$, we say that μ is a *maximising measure* for f. We denote the set of all maximising measures for f by $\mathcal{M}_{\max}(f)$. The study of maximising measures is a topic of substantial recent research interest: for further information on this topic we direct the reader to the survey article of O. Jenkinson.[4]

For the purposes of this article, we will define the *pressure* of a continuous function $f\colon \Sigma_A \to \mathbb{R}$ to be the quantity

$$P(f) = \sup \left\{ h(\nu) + \int f \, d\nu \colon \nu \in \mathcal{M}_\sigma \text{ and } \int f \, d\nu \neq -\infty \right\}, \tag{1}$$

where $h(\nu)$ denotes the entropy of the measure ν (see e.g. (Ref. 13)). Some alternative formulations of the pressure may be found in (Ref. 7, Sec. 2).

We say that a measure $\mu \in \mathcal{M}_\sigma$ is an *equilibrium state* for f if $P(f) = h(\mu) + \int f \, d\mu$, and say that μ is an *invariant Gibbs measure* for f if $\mu \in \mathcal{M}_\sigma$ and there is a real constant $Q > 1$ such that

$$Q^{-1} \leq \frac{\mu\left([x_1, \ldots, x_n]\right)}{e^{S_n f(x) - nP(f)}} \leq Q$$

for every $n > 0$ and $x \in \Sigma_A$.

When $\mathcal{I}$ is finite and σ is topologically mixing, it is well known that every Hölder continuous f possesses a unique invariant Gibbs measure, and that this measure is also the unique equilibrium state for f[9,10]. When $\mathcal{I}$ is infinite, it may be the case that certain Hölder continuous functions fail to have invariant Gibbs states. To ensure the existence of invariant Gibbs states we impose the following additional condition on f.

Definition 2.1.

We call a continuous function $f\colon \Sigma_A \to \mathbb{R}$ *summable* if it satisfies the condition

$$\sum_{i \in \mathcal{I}} \exp\left(\sup_{x \in [i]} f(x) \right) < \infty.$$

We write $\mathcal{S}_A$ for the set of all summable functions $f\colon \Sigma_A \to \mathbb{R}$ which satisfy the additional condition $|f|_{\text{var}} < \infty$.

In the case where $\mathcal{I}$ is infinite, the existence of invariant Gibbs states requires a further condition on A. We say that the matrix A is *finitely primitive* if there exist an integer N and a finite set $\mathcal{J} \subseteq \mathcal{I}$ such that for each $a, b \in \mathcal{I}$ we can find

$(i_1, \ldots, i_N) \in \mathcal{J}^N$ such that the cylinder set $[a, i_1, \ldots, i_N, b] \subseteq \Sigma_A$ is nonempty. R. D. Mauldin and M. Urbański proved the following theorem (see (Ref. 7 and 8)):

Theorem 1. (Mauldin-Urbański) *If $g \in \mathcal{S}_A$ and A is finitely primitive, then $P(g)$ is finite and g has a unique invariant Gibbs measure μ_g. This measure satisfies*

$$e^{-4|g|_{\mathrm{var}}} \leq \frac{\mu_g([x_1, \ldots, x_n])}{e^{S_n g(x) - nP(g)}} \leq e^{4|g|_{\mathrm{var}}} \tag{2}$$

for every $x \in \Sigma_A$ and $n > 0$.

In fact, the above theorem was proved under the stronger assumption that $\mathrm{var}_n g = O(\theta^n)$ for some $0 < \theta < 1$, but as remarked in (Ref. 6) the proof generalises without difficulty. The requirement of finite primitivity was later shown by O. Sarig to be necessary as well as sufficient[12].

It should be noted that while every $g \in \mathcal{S}_A$ has finite pressure and possesses an invariant Gibbs state, in some cases this invariant Gibbs state may have infinite entropy, and hence is not an equilibrium state in the sense defined above. However, Mauldin and Urbański were able to show the following:

Proposition 1. *Let $g \in \mathcal{S}_A$ and suppose that the additional condition*

$$\sum_i \sup_{x \in [i]} |g(x)| \exp\left(\sup_{x \in [i]} g(x)\right) < \infty \tag{3}$$

holds. Then the invariant measure μ_g satisfies $h(\mu_g) < \infty$ and $\int |g|\, d\mu_g < \infty$, and μ_g is an equilibrium state for g in the sense that $h(\mu_g) + \int g\, d\mu_g = P(g)$.

We will see in the following section that if $f \in \mathcal{S}_A$, then for each $t > 1$ the function tf satisfies condition (3) above automatically. Hence the map $t \mapsto \mu_{tf}$ which will be studied in this article is well-defined, and for every $t > 1$ the measure μ_{tf} has finite entropy and is both an equilibrium state and a Gibbs state.

O. Jenkinson, R.D. Mauldin and M. Urbański showed in (Ref. 6) that the family of measures $(\mu_{tf})_{t \geq 1}$ has at least one accumulation point in $\mathcal{M}_\sigma$, and showed that every accumulation point must be a maximising measure for f. They then asked whether the entropy of these accumulation points could be characterised either as the supremum of the entropies of the maximising measures, or as the limit of the entropies $h(\mu_{tf})$. In this note we prove the following:

Theorem 1. *Let $\sigma \colon \Sigma_A \to \Sigma_A$ be a finitely primitive subshift of finite type, and let $f \in \mathcal{S}_A$. For each $t > 1$, let μ_{tf} be the unique Gibbs equilibrium state associated* *to the potential tf, and let μ_∞ be any accumulation point at infinity of $(\mu_{tf})_{t\geq 1}$. Then,*

$$h(\mu_\infty) = \lim_{t\to\infty} P(tf - t\beta(f)) = \lim_{t\to\infty} h(\mu_{tf}) = \sup_{\nu\in\mathcal{M}_{\max}(f)} h(\nu).$$

This theorem constitutes a positive answer to Questions 1 and 2 in (Ref. 6).

3. PROOF OF THEOREM 1

It is convenient to break up the proof of Theorem 1 into several separate statements. In particular, the theorem is obtained by combining Lemmata 3.1 and 3.5 below. We include some observations from (Ref. 6).

Throughout this section we assume that the matrix A is finitely primitive and that $f \in \mathcal{S}_A$.

Lemma 3.1. *For each $t > 1$ we have*

$$\sum_i \sup_{x\in[i]} |tf(x)| \exp\left(\sup_{x\in[i]} tf(x)\right) < \infty,$$

so that the conclusions of Proposition 1 apply to μ_{tf}.

Proof: Let $t > 1$. Since $f \in \mathcal{S}_A$ it is clear that there exists $C_t > 0$ such that

$$\sup_i \sup_{x\in[i]} |tf(x)|\, e^{(t-1)\sup_{x\in[i]} f(x)} \leq C_t.$$

Hence,

$$\sum_i \sup_{x\in[i]} |tf(x)| \exp\left(\sup_{x\in[i]} tf(x)\right) \leq C_t \sum_i \exp\left(\sup_{x\in[i]} f(x)\right) < \infty.$$

Lemma 3.2. *The maps $t \mapsto h(\mu_{tf})$ and $t \mapsto P(tf - t\beta(f))$ are decreasing and bounded below on the interval $(1,\infty)$, and satisfy*

$$\lim_{t\to\infty} h(\mu_{tf}) = \lim_{t\to\infty} P(tf - t\beta(f)) \geq \sup_{\nu\in\mathcal{M}_{\max}(f)} h(\nu).$$

Proof: Recall from (Ref. 7, Sec. 2) that the map $t \mapsto P(tf)$ is analytic, and that its first and second derivatives admit the characterisations

$$P'(tf) = \int f\, d\mu_{tf} \leq \beta(f)$$

and

$$P''(tf) = \lim_{n\to\infty} \frac{1}{n} \int \left(S_n \left[f - \int f \, d\mu_{tf} \right] \right)^2 d\mu_{tf} \geq 0.$$

Note also that $P(tf - t\beta(f)) = P(tf) - t\beta(f)$ as a consequence of (1). We have $P'(tf - t\beta(f)) = P'(tf) - \beta(f) = \int f \, d\mu_{tf} - \beta(f) \leq 0$ for every $t > 1$, and so $P(tf - t\beta(f))$ is decreasing. To see that $t \mapsto h(\mu_{tf})$ is decreasing we note that for every $t > 1$,

$$h(\mu_{tf}) = P(tf) - tP'(tf)$$

so that

$$h'(\mu_{tf}) = -tP''(tf) \leq 0.$$

Clearly $h(\mu_{tf}) \geq 0$ for all $t > 1$, and $P(tf - t\beta(f)) \geq 0$ by the variational principle (1). Since both maps are decreasing, we conclude that $\lim_{t\to\infty} h(\mu_{tf})$ and $\lim_{t\to\infty} P(tf - t\beta(f))$ both exist. It follows that the limit

$$\lim_{t\to\infty} tP'(tf) - t\beta(f) = \lim_{t\to\infty} (P(tf - t\beta(f)) - h(\mu_{tf}))$$

exists also. Since $P(tf - t\beta(f))$ is convergent and monotone, the integral $\int_{t=2}^{\infty} \left| P'(tf) - \beta(f) \right| dt$ must be finite; it follows that $\lim_{t\to\infty} tP'(tf) - t\beta(f)$ cannot be nonzero, and so

$$\lim_{t\to\infty} h(\mu_{tf}) = \lim_{t\to\infty} P(tf - t\beta(f))$$

as required. To complete the proof, we observe that $P(tf - t\beta(f)) \geq \sup_{\nu\in\mathcal{M}_{max}(f)} h(\nu)$ for all $t > 1$ as a consequence of the variational principle (1).

Lemma 3.3. *Let $f \in \mathcal{S}_A$. Then,*

$$\lim_{t\to\infty} t^{-1} P(tf) = \beta(f).$$

Proof: This follows from Lemma 3.2 and the relation $P(tf) = P(tf - t\beta(f)) + t\beta(f)$.

Lemma 3.4. *Let μ_∞ be an accumulation point at infinity of $(\mu_{tf})_{t>1}$ in the weak-* topology. Then μ_∞ is a probability measure and is maximising for f.*

Proof: See (Ref. 6, Theorem 1).

Lemma 3.5. *Let $\mu_\infty = \lim_{k\to\infty} \mu_{t_k f}$ be an accumulation point at infinity of $(\mu_{tf})_{t>1}$ in the weak-* topology. Then,*

$$h(\mu_\infty) = \lim_{t\to\infty} h(\mu_{tf}) = \sup_{\nu\in\mathcal{M}_{\max}(f)} h(\nu).$$

Proof: By Lemmata 3.2 and 3.4 we have

$$h(\mu_\infty) \le \sup_{\nu\in\mathcal{M}_{\max}(f)} h(\nu) \le \lim_{t\to\infty} h(\mu_{tf}),$$

and so it suffices to establish

$$\lim_{t\to\infty} h(\mu_{tf}) \le h(\mu_\infty).$$

For $\mathcal{K} \subseteq \mathcal{I}$ and $n > 0$, we use the symbol $\hat{\mathcal{K}}^n$ to denote set of all n-tuples $(i_1, \ldots, i_n) \in \mathcal{K}^n$ such that the cylinder $[i_1, \ldots, i_n] \subseteq \Sigma_A$ is nonempty. For each $n > 0$ we define

$$\mathcal{A}^n = \{[i_1, \ldots, i_n] \subset \Sigma_A : (i_1, \ldots, i_n) \in \hat{\mathcal{I}}^n\}.$$

Clearly each $\mathcal{A}^n$ is a countable measurable partition of Σ_A.

For $x \in [0, 1]$ define $\phi(x) = -x \log x$ when $x \neq 0$, and $\phi(0) = 0$; note that ϕ is continuous. Given $\nu \in \mathcal{M}_\sigma$ and $n > 0$, we let

$$H(\nu \mid \mathcal{A}^n) = \sum_{(i_1,\ldots,i_n)\in\mathcal{I}^n} \phi\left(\nu\left([i_1, \ldots, i_n]\right)\right).$$

Using standard facts from entropy theory, the entropy $h(\nu)$ of an invariant measure $\nu \in \mathcal{M}_\sigma$ is equal to $h(\nu) = \inf_{n\ge1} \frac{1}{n} H(\nu \mid \mathcal{A}^n)$. We claim that for each $n > 0$,

$$\lim_{k\to\infty} H\left(\mu_{t_k f} \mid \mathcal{A}^n\right) = H\left(\mu_\infty \mid \mathcal{A}^n\right). \tag{4}$$

Since $\sup_{x\in[i]} f(x) \to -\infty$ as $i \to \infty$, we may choose a finite set $\mathcal{J} \subset \mathcal{I}$ such that

$$4|f|_{\text{var}} - n\beta(f) + 1 + (n-1)\sup f + \sup_{i\in\mathcal{I}\setminus\mathcal{J}} \sup_{x\in[i]} f(x) < 0 \tag{5}$$

and

$$4|f|_{\text{var}} + n\beta(f) + 1 \le \sup_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} \sup_{x\in[i_1,\ldots,i_n]} |S_n f(x)| \tag{6}$$

and

$$\sup_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} \left(\sup_{x\in[i_1,\ldots,i_n]} |S_n f(x)|\, e^{\sup_{x\in[i_1,\ldots,i_n]} S_n f(x)} \right) \le 1. \tag{7}$$

It is then easily seen that as $t \to \infty$,

$$\left(te^{4t|f|_{\text{var}}-nt\beta(f)+t}\sum_{i\in\mathcal{I}\setminus\mathcal{J}} e^{(t-1)\sup_{x\in[i]} f(x)}\right)\left(\sum_{i\in\mathcal{I}} e^{(t-1)\sup_{x\in[i]} f(x)}\right)^{n-1} \to 0,$$

as a consequence of (5) above. By Lemma 3.3, we may choose $T > 2$ such that $t > T$ implies

$$tn\beta(f) - t \le nP(tf) \le tn\beta(f) + t. \tag{8}$$

Let $t > T$ and $(i_1, \ldots, i_n) \in \hat{\mathcal{I}}^n \setminus \hat{\mathcal{J}}^n$. Applying inequalities (2), (6) and (8), we obtain

$$\begin{aligned}
-\log\left(\mu_{tf}([i_1, \ldots, i_n])\right) &\le \left(4t|f|_{\text{var}} + nP(tf) - t\sup_{x\in[i_1,\ldots,i_n]} S_n f(x)\right)\\
&\le t\left(4|f|_{\text{var}} + n\beta(f) + 1 + \sup_{x\in[i_1,\ldots,i_n]} |S_n f(x)|\right)\\
&\le 2t\sup_{x\in[i_1,\ldots,i_n]} |S_n f(x)|.
\end{aligned}$$

Combining this with inequalities (2), (7) and (8), we deduce that for every $t > T$,

$$\begin{aligned}
&\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} \phi(\mu_{tf}([i_1, \ldots, i_n]))\\
&= \sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} -\log\mu_{tf}([i_1, \ldots, i_n])\,\mu_{tf}([i_1, \ldots, i_n])\\
&\le 2t\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} \sup_{x\in[i_1,\ldots,i_n]} |S_n f(x)|\,\mu_{tf}([i_1, \ldots, i_n])\\
&\le 2te^{4t|f|_{\text{var}}-nP(tf)}\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} \sup_{x\in[i_1,..,i_n]} |S_n f(x)|\,e^{t\sup_{x\in[i_1,..,i_n]} S_n f(x)}\\
&\le 2te^{4t|f|_{\text{var}}-tn\beta(f)+t}\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n} e^{(t-1)\sup_{x\in[i_1,\ldots,i_n]} S_n f(x)}\\
&\le 2nt\left(e^{4t|f|_{\text{var}}-tn\beta(f)+t}\sum_{i\in\mathcal{I}\setminus\mathcal{J}} e^{(t-1)\sup_{x\in[i]} f(x)}\right)\left(\sum_{i\in\mathcal{I}} e^{(t-1)\sup_{x\in[i]} f(x)}\right)^{n-1}
\end{aligned}$$

so that

$$\lim_{t\to\infty}\left(\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n}\phi\left(\mu_{tf}\left([i_1,\ldots,i_n]\right)\right)\right)=0. \tag{9}$$

A similar, simpler argument using inequalities (2), (5) and (8) shows that when $(i_1,\ldots,i_n)\in\hat{\mathcal{I}}^n\setminus\hat{\mathcal{J}}^n$, one has

$$\mu_\infty\left([i_1,\ldots,i_n]\right)=0,$$

from which it follows that

$$H(\mu_\infty|\mathcal{A}^n)=\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{J}}^n}\phi\left(\mu_\infty\left([i_1,\ldots,i_n]\right)\right).$$

Combining this with (9) above, we deduce that to prove (4) we need only show that

$$\lim_{k\to\infty}\left(\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{J}}^n}\phi(\mu_{t_kf}([i_1,\ldots,i_n]))\right)=\sum_{(i_1,\ldots,i_n)\in\hat{\mathcal{J}}^n}\phi(\mu_\infty([i_1,\ldots,i_n])).$$

By weak-* convergence one has $\lim_{k\to\infty}\mu_{t_kf}([i_1,\ldots,i_n])=\mu_\infty([i_1,\ldots,i_n])$ for all cylinder sets; since ϕ is continuous and the sum is finite, the required convergence follows. The claim (4) is proved.

To complete the proof of the Lemma, we suppose for a contradiction that $h(\mu_\infty)<\lim_{t\to\infty}h(\mu_{tf})$. Choose $\varepsilon>0$ such that $h(\mu_\infty)\le\lim_{t\to\infty}h(\mu_{tf})-3\varepsilon$, and choose $N>0$ such that $\frac{1}{N}H\left(\mu_\infty|\mathcal{A}^N\right)\le h\left(\mu_\infty\right)+\varepsilon$. Using (4), we deduce that for all sufficiently large $k>0$,

$$\begin{aligned}h(\mu_{t_kf})&\le\frac{1}{N}H\left(\mu_{t_kf}|\mathcal{A}^N\right)\le\frac{1}{N}H\left(\mu_\infty|\mathcal{A}^N\right)+\varepsilon\\&\le h(\mu_\infty)+2\varepsilon\le\lim_{t\to\infty}h(\mu_{tf})-\varepsilon.\end{aligned}$$

Taking the limit as $k\to\infty$ yields the required contradiction.

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REFERENCES

1. Z. Coelho, Entropy and ergodicity of skew-products over subshifts of finite type and central limit asymptotics, PhD thesis, Warwick University (1990).
2. J.-P. Conze and Y. Guivarc'h, Croissance des sommes ergodiques, unpublished manuscript (*c.* 1993).
3. G. Contreras, A. Lopes and P. Thieullen, Lyapunov minimizing measures for expanding maps of the circle, *Ergod. Theory Dynam. Systems* **5**:1379–1409 (2001).
4. O. Jenkinson, Ergodic optimization, *Discrete Contin. Dyn. Syst.* **15**:197–224 (2006).
5. O. Jenkinson, Geometric barycentres of invariant measures for circle maps, *Ergod. Theory Dynam. Systems* **21**:511–532 (2001).
6. O. Jenkinson, R.D. Mauldin and M. Urbański, Zero temperature limits of Gibbs-equilibrium states for countable alphabet subshifts of finite type, *J. Stat. Phys.* **119**:765–776 (2005).
7. R.D. Mauldin and M. Urbański, *Graph directed Markov systems: geometry and dynamics of limit sets*, Cambridge Tracts in Mathematics 148, Cambridge University Press 2003.
8. R. D. Mauldin and M. Urbański, Gibbs states on the symbolic space over an infinite alphabet, *Israel J. Math.* **125**:93–130 (2001).
9. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque No. 187–188 (1990).
10. D. Ruelle, *Thermodynamic formalism: the mathematical structures of equilibrium statistical mechanics*, 2nd edition, Cambridge University Press, 2004.
11. O. Sarig, Thermodynamic formalism for countable Markov shifts, *Ergod. Theory Dynam. Systems* **19**:1565–1593 (1999).
12. O. Sarig, Characterization of existence of Gibbs measures for countable Markov shifts, *Proc. Amer. Math. Soc.* **131**:1751–1758 (2003).
13. P. Walters, *An introduction to ergodic theory* (Springer, Berlin, 1982).